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Closedness in the Weak Topology of the Dual Pair L^1, $\mathscr{C}$

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This paper gives improvements and new results along the lines of the work of Rockafellar, Olech, and Sainte-Beuve. It is concerned with closure of sets and lower semicontinuity of functionals relative to the weak topologies on L^1 or on the space $\mathscr{M}$ of measures.

1. INTRODUCTION

This paper is in the line of Rockafellar [11], Olech [6, 7], and Sainte-Beuve [13, 17].

Let T be a topological space, μ a positive measure on T, and E a separable Fréchet space. Let Γ be a measurable multifunction from T to E', the dual space of E. We denote by $\mathscr{L}_\Gamma^1$ the set of μ-integrable selections of Γ, by L_Γ^1 its quotient. Let $\mathscr{C}_E{}^b$ be a vector space of E-valued continuous bounded functions on T. Consider the geometric problems: What is the closure of L_Γ^1 for the weak topology $\sigma(L^1_{E'}, \mathscr{C}_E{}^b)$, what is the weak closure of L_Γ^1 in the space $\mathscr{M}_{E'}$ of E'-valued measure? Now let $f: T \times E' \to \bar{\mathbf{R}}$ be a measurable function and define $I_f(u)$ for $u \in L_E{}^1$, by $I_f(u) = \int f(t, u(t))\, \mu(dt)$. Consider the functional problems: What is the weak l.s.c. hull of I_f, what is the polar (calculated on $\mathscr{M}_{E'}$) of its polar?

First Rockafellar in [11] studied the functional problem and gave corollaries about the geometric problem. His work is an application of his results on the duality between L^∞ and $L^{\infty\prime}$. Then Olech [6] treated the geometric problem by a completely different approach. In [7] he used [6] to prove that I_f is l.s.c. iff L_Γ^1 is closed (where $\Gamma(t)$ is the epigraph of $f(t, \cdot)$). Sainte-Beuve in [13] weakens the hypotheses of [6]: T becomes a completely regular Suslin space instead of compact metric, and E becomes a separable Fréchet space instead of $\mathbf{R}^n$. In [17] Sainte-Beuve extends [7].

Applications of those kinds of results are given in Olech [8, 9]. Some other similar problems have been recently studied in Brézis [1], Ioffe [3, 4], and Tran cao Nguyen [20].

In Section 2 we set notation and hypotheses. In Section 3 we state briefly the foregoing results we need. In Section 4 we give some useful examples. In

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Section 5 we give our results about the geometric problem. We extend a necessary and sufficient condition for $L_\Gamma{}^1$ to be closed (Olech [6, Corollary 2, p. 317]) within the framework of Sainte-Beuve. Then we give a new formula for the closure of $L_\Gamma{}^1$. In Section 6 we apply our geometric results to characterize the bipolar of I_f. The formula is similar to [11, p. 458]. The main improvements are: (1) Rockafellar needs hypotheses which ensure that I_f is l.s.c., (2) he assumes $E = \mathbf{R}^n$, and (3) he deduces the formula from a rather technical preliminary result [11, Theorem 4, p. 452].

Nevertheless in [11] T is not assumed to be separable. Then we give necessary and sufficient conditions for I_f to be l.s.c. and a description of its l.s.c. hull.

All the results of Section 5 (except 3 of Corollary 4) are valid when T is a completely regular Suslin space and μ is σ-finite. But, in Section 6 we suppose that T is compact. To avoid changing the hypotheses and notation in the middle of the text, we assume throughout the paper that T is compact. One can find in [22] the general version; the proofs are identical. Finally we wish to thank M. F. Nougues-Sainte-Beuve for a very helpful conversation.

2. Notations and Hypotheses

Let T be a compact metric space, μ a positive bounded measure on T, and E a separable Fréchet space. We denote by $\mathscr{C}_E(T)$ (or shortly $\mathscr{C}_E$, even $\mathscr{C}$ when $E = \mathbf{R}$) the space of continuous E-valued functions on T. When it is endowed with the uniform topology its dual identifies with a subset of the set of E'-valued measures.

We denote by $\mathscr{B}(E_s')$ the Borel tribe of the $\sigma(E', E)$ topology, by $\mathscr{B}(T)$ the Borel tribe of T, and by $\mathscr{B}(T)_\mu$ its μ-completion. It is well known that E_s' is a Lusin (or standard) space, hence, a Suslin space, so that one can define μ-measurability of a function $u\colon T \to E_s'$ in any of the classical ways (see [2, Theorem III.36, p. 83]).

For any positive measure θ on T we denote by $\mathscr{L}_{E'}^1([E], \theta)$ the space of all θ-measurable functions $u\colon T \to E_s'$ such that there exist $\alpha \in \mathscr{L}_{\mathbb{R}}{}^1(\theta)$ and an equicontinuous set $Q \subset E'$ such that $u(t) \in \alpha(t)\, Q$ θ-a.e. And $L_{E'}^1([E], \theta)$ denotes the quotient space. We shall write $\mathscr{L}_{E'}^1$ and $L_{E'}^1$ for $\mathscr{L}_{E'}^1([E], \mu)$ and $L_{E'}^1([E], \mu)$.

For every θ, and every $u \in \mathscr{L}_{E'}^1([E], \theta)$ the function u is θ-Pettis integrable, so that the formula

$$m(A) = \int_A u \, d\theta$$

defines an E'-valued measure. It is denoted by $u \cdot \theta$. The set of all measures $u \cdot \theta$ for $u \in \mathscr{L}_{E'}^1([E], \theta)$ is by definition [13] the set of E'-valued measures with bounded variation and absolutely continuous with respect to θ. The space of all vector-valued measures with bounded variation (for all θ) is denoted by $\mathscr{M}_{E'}(T)$

or shortly $\mathscr{M}_{E'}$. It is the dual space of $\mathscr{C}_E$. So we have at least three pairs of spaces in separated duality: E and E', $\mathscr{C}_E$ and $L^1_{E'}$, $\mathscr{C}_E$ and $\mathscr{M}_{E'}$. Let C be a subset of one of these spaces, then $\delta^*(\cdot \mid C)$ denotes its support function and C^0 its polar: for example, if $C \subset E'$

$$\delta^*(x \mid C) = \sup\{\langle x', x\rangle \mid x' \in C\} \qquad \text{for } x \in E,$$

$C^0 = \{x \in E \mid \forall x' \in C \langle x, x'\rangle \leqslant 1\}$. And co C denotes the convex hull of C. If C is closed and convex, then As C denotes the asymptotic cone of C [2, I.7, p. 8], and C is said to be LCNL is it is weakly locally compact and contains no line [2, I.15, p. 16–17).

Let Γ be a multifunction from T to E'. We say that it is μ-measurable if its graph belongs to $\mathscr{B}(T)_\mu \otimes \mathscr{B}(E'_s)$.

We denote by $\mathscr{L}_\Gamma{}^1$ the set $\{u \in \mathscr{L}^1_{E'} \mid u(t) \in \Gamma(t)\ \mu \text{ a.e.}\}$ and by $L_\Gamma{}^1$ its quotient. We denote by $\overline{L_\Gamma{}^1}$ the $\sigma(L^1_{E'}, \mathscr{C}_E)$ closure of $L_\Gamma{}^1$ and by $\overline{\overline{L_\Gamma{}^1}}$ its $\sigma(\mathscr{M}_{E'}, \mathscr{C}_E)$ closure. We shall denote by $\mathscr{H}$ the conic convex set dom $\delta^*(\cdot \mid L_\Gamma{}^1) = \{\varphi \in \mathscr{C}_E \mid \delta^*(\varphi \mid L_\Gamma{}^1) < \infty\}$.

Hypotheses

(H1) μ is nonatomic or $\Gamma(t)$ is μ a.e. convex. (This hypothesis could be weakened as in [12]: Γ is supposed convex valued on every atom of μ.)

(H2) $L_\Gamma{}^1$ is nonempty.

3. Foregoing Results

The set $\overline{\overline{L_\Gamma{}^1}}$ has two important properties: (1) It is convex, thanks to the Liapunov theorem, (2) it is the intersection of a countable family of closed half spaces. This follows from the fact that $\mathscr{M}_{E'}$ has the Lindelöf property. (When E is Fréchet and T completely regular Suslin that result has been proved by Sainte-Beuve.) So there exists a sequence (φ_n) in $\mathscr{C}_E$ such that $\forall n$, $\delta^*(\varphi_n \mid L_\Gamma{}^1) < \infty$ and

$$\overline{\overline{L_E{}^1}} = \{m \in \mathscr{M}_{E'} \mid \forall n, \langle m, \varphi_n\rangle \leqslant \delta^*(\varphi_n \mid L_E{}^1)\}.$$

We can now state the Olech theorem extended by Sainte-Beuve:

THEOREM 0. *Putting*

$$\Sigma(t) = \{x' \mid \forall n, \langle x', \varphi_n(t)\rangle \leqslant \delta^*(\varphi_n(t) \mid \Gamma(t))\}$$
$$C(t) = \{x' \mid \forall n, \langle x', \varphi_n(t)\rangle \leqslant 0\},$$

one has

$$\overline{L_\Gamma{}^1} = L_\Sigma{}^1$$

and

$$\overline{\overline{L_E^1}} = L_\Sigma^1 + \{m \mid \exists\theta, \exists u \in L_C^1(\theta) \text{ such that } m = u \cdot \theta\}.$$

In the last formula one can consider only measures θ which are μ-singular. Moreover $\forall t$, $C(t) = [\mathscr{H}(t)]^0$ (where $\mathscr{H}(t) = \{\varphi(t) \mid \varphi \in \mathscr{H}\}$).

Remark 1. It is easy to see that $\mathscr{H}(t)$ and $C(t)$ depend only on the values of Γ on a neighborhood of t. Indeed: $\mathscr{H}(t) = \{\varphi(t) \mid \varphi \in \mathscr{C}_E, \exists V$ open such that $t \in V$ and $\int_V \delta^*(\varphi(s) \mid \Gamma(s))\, \mu(ds) < \infty\}$, because if φ belongs to the right-hand side, then there exists a continuous function $\alpha\colon T \to [0, 1]$ such that $\alpha(t) = 1$ and $\alpha(s) = 0$ if $s \notin V$. Then $\alpha\varphi \in \mathscr{H}$ and $\varphi(t) = (\alpha\varphi)(t)$ belongs to $\mathscr{H}(t)$.

Remark 2. It is obvious from the definition of Σ and C that $\Sigma(t) \neq \phi$ and $\delta^*(\varphi_n(t) \mid \Gamma(t)) < \infty$ implies $C(t) = \operatorname{As} \Sigma(t)$. As L_Γ^1 is nonempty, L_Σ^1 is also nonempty and so $\Sigma(t) \neq \phi$ a.e. Hence one has $C(t) = \operatorname{As} \Sigma(t)$ a.e.

THEOREM 1. *The set L_Γ^1 is $\sigma(L^1_{E'}, \mathscr{C}_E)$ closed iff there exist two sequences (φ_n) in $\mathscr{C}_E$ and (ψ_n) in $\mathscr{L}_{\mathbb{R}}^1$ such that*

$$\Gamma(t) = \{x' \mid \forall n, \langle x', \varphi_n(t)\rangle \leqslant \psi_n(t)\}\ \mu \text{ a.e.} \tag{1}$$

Remark 3. The above necessary and sufficient condition is not very practical. Olech gave also another one, which we extend in Theorem 3.

Remark 4. The relation $\psi_n \in \mathscr{L}_{\mathbb{R}}^1$ is essential in the proof of sufficiency: see Example 1 below.

4. EXAMPLES

EXAMPLE 1. Let $T = [0, 1]$, $\mu = dt + \delta_0$ where dt is the Lebesgue measure and δ_0 the Dirac measure, $E = \mathbf{R}$ and

$$\begin{aligned} \Gamma(t) &= \{0\} && \text{if} \quad t = 0 \\ &= [0, 1/t] && \text{if} \quad t > 0. \end{aligned}$$

Then for $n \geqslant 1$ the function $u_n = n\chi_{]0,1/n]}$ belongs to $\mathscr{L}_\Gamma^1$, but it converges weakly to $\chi_{\{0\}}$ which does not belong to $\mathscr{L}_\Gamma^1$. It will be seen after Theorem 3 that $\overline{L_\Gamma^1} = L_\Sigma^1$ with

$$\begin{aligned} \Sigma(t) &= [0, \infty[&& \text{if} \quad t = 0 \\ &= [0, 1/t] && \text{if} \quad t > 0. \end{aligned}$$

Finally we notice that (1) is satisfied with a nonintegrable ψ_n:

$$\Gamma(t) = \{x' \mid \langle x', \varphi_1(t)\rangle \leqslant \psi_1(t) \text{ and } \langle x', \varphi_2(t)\rangle \leqslant \psi_2(t)\}$$

with

$$\begin{aligned} \varphi_1 = -1, \qquad \psi_1 &= 0, \qquad \varphi_2 = 1, \\ \psi_2(t) &= 0 \qquad \text{if} \qquad t = 0 \\ &= 1/t \qquad \text{if} \qquad t > 0. \end{aligned}$$

EXAMPLE 1′. In Example 1 μ has an atom. We can introduce an auxiliary variable: Let $T = [0, 1]^2$, $\mu = (dt + \delta_0) \otimes ds$,

$$\begin{aligned} \Gamma(t, s) &= \{0\} \qquad \text{if} \qquad t = 0 \\ &= [0, 1/t] \qquad \text{if} \qquad t > 0. \end{aligned}$$

Thus μ is nonatomic but the same phenomena happens.

EXAMPLE 2. Let $T = [0, 1]$, $\mu = dt$, $E = \mathbf{R}$, and let K be an analogous of the Cantor set but with positive measure. Let

$$\begin{aligned} \Gamma(t) &= \{0\} \qquad \text{if} \qquad t \in K \\ &= [0, \infty[\qquad \text{if} \qquad t \in T - K. \end{aligned}$$

It will be proved after Theorem 3 that $\overline{L_\Gamma{}^1} = L^1_{[0,\infty[}$. (A similar example is given in [8, p. 622].)

Remark 5. One could think that a necessary condition (which would fail in the examples) for $L_\Gamma{}^1$ to be closed is a kind of semicontinuity of Γ. This is not true in an elementary way: Indeed if a, b belong to $\mathscr{L}_{\mathbb{R}}{}^1$ with $a \leqslant b$, and $\Gamma(t) = [a(t), b(t)]$, it follows from Theorem 1 that $L_\Gamma{}^1$ is closed (it is even $\sigma(L^1, L^\infty)$ compact!) whatever a and b are. The best idea is to consider the multifunction $t \mapsto \operatorname{dom} \delta^*(\cdot \mid \Gamma(t))$. That will be stated precisely in Corollary 4.

5. GEOMETRIC RESULTS

The following result is an easy consequence of von Neumann–Aumann–Sainte–Beuve theorems [2, III.22, p. 74, III.23, p. 75].

LEMMA 2. *If Γ and Γ' are μ-measurable multifunctions such that $L_\Gamma{}^1 = L^1_{\Gamma'} \neq \phi$ then $\Gamma(t) = \Gamma'(t)$ μ-a.e.*

THEOREM 3. (1) *A necessary condition for $L_\Gamma{}^1$ to be closed in $L^1_{E'}$, is: $\Gamma(t)$ is* a.e. *closed and convex, and* As $\Gamma(t) = [\mathscr{H}(t)]^0$ a.e.

(2) *Suppose that* $\dim E < \infty$ *or Γ is* LCNL *valued. Then a sufficient condition for $L_\Gamma{}^1$ to be closed is:*

$$\Gamma(t) \qquad \textit{is a.e. closed and convex, and} \qquad \text{As}\, \Gamma(t) \supset [\mathscr{H}(t)]^0 \text{ a.e.}$$

(3) *Suppose that* $\dim E < \infty$ *or* $\Gamma(t)$ *is contained in a fixed* LCNL *subset of* E'. *Then* $\overline{L_\Gamma^1} = L_\Sigma^1$ *with*

$$\Sigma(t) = \overline{\operatorname{co} \Gamma(t) + [\mathscr{H}(t)]^0}.$$

Proof. (1) By Theorem 0 $\overline{L_\Gamma^1} = L_\Sigma^1$, and by Lemma 2 $\Gamma(t) = \Sigma(t)$ a.e. We noticed in Remark 2 that $C(t) = \operatorname{As} \Sigma(t)$ a.e. Then $C(t) = \operatorname{As} \Gamma(t)$ a.e.

(2) Let (φ_n) be the sequence of Theorem 0. Let (ψ_k) be the countably family of all $\sum_{i=0}^{p} \alpha_i \varphi_i$ for $p \in \mathbf{N}$, $\alpha_i \in \mathbf{Q}_+$. As $\mathscr{H}$ is conic, one has $\psi_k \in \mathscr{H}$. Then one has

$$\begin{aligned} C(t) &= \{x' \mid \forall n, \langle x', \varphi_n(t)\rangle \leqslant 0\} \\ &= \{x' \mid \forall k, \langle x', \psi_k(t)\rangle \leqslant 0\}. \end{aligned}$$

As $\overline{\{\psi_k(t) \mid k \in \mathbf{N}\}}$ is conic we have

$$C(t) = [\overline{\{\psi_k(t) \mid k \in \mathbf{N}\}}]^0 \qquad \text{a.e.} \tag{2}$$

Moreover a.e.

$$\begin{aligned} \operatorname{As} \Gamma(t) &\supset [\mathscr{H}(t)]^0 \qquad &&\text{(by hypothesis)} \\ &= C(t) \\ &\supset \operatorname{As} \Gamma(t) &&\text{(because } \Sigma \supset \Gamma) \end{aligned}$$

hence

$$C(t) = \operatorname{As} \Gamma(t) \qquad \text{a.e.} \tag{3}$$

From [2, I.7, p. 8]

$$\operatorname{As} \Gamma(t) = [\operatorname{dom} \delta^*(\cdot \mid \Gamma(t))]^0. \tag{4}$$

Combining (2), (3), (4) we obtain

$$[\operatorname{dom} \delta^*(\cdot \mid \Gamma(t))]^0 = [\overline{\{\psi_k(t) \mid k \in \mathbf{N}\}}]^0 \qquad \text{a.e.}$$

hence

$$\overline{\operatorname{dom} \delta^*(\cdot \mid \Gamma(t))} = \overline{\{\psi_k(t) \mid k \in \mathbf{N}\}} \qquad \text{a.e.} \tag{5}$$

We set $\Theta(t) = \{x' \mid \forall k, \langle x', \psi_k(t)\rangle \leqslant \delta^*(\psi_k(t) \mid \Gamma(t))\}$. By Theorem 1 L_Θ^1 is closed. Moreover it is obvious that $\Gamma(t) \subset \Theta(t) \subset \Sigma(t)$ a.e. hence $\overline{L_\Gamma^1} = L_\Theta^1$. It remains to show that $\Gamma(t) = \Theta(t)$ a.e.

(a) First suppose that $\Gamma(t)$ is LCNL. By [2, I.15] $\operatorname{dom} \delta^*(\cdot \mid \Gamma(t))$ has a nonempty interior. So by (5) the $\psi_k(t)$ which belong to $\operatorname{dom} \delta^*(\cdot \mid \Gamma(t))$ are dense in it. And by [2, III.34, p. 82], $\Gamma(t) = \Theta(t)$.

(b) Suppose now $\dim E < \infty$. The two cones in (5) are contained in the subspace generated by $\operatorname{dom} \delta^*(\cdot \mid \Gamma(t))$. Denote by $A(t)$ the relative interior of $\operatorname{dom} \delta^*(\cdot \mid \Gamma(t))$. By (5) the $\psi_k(t)$ which belong to $A(t)$ are dense in it. By

Rockafellar [10, Theorem 10.1, p. 82] $\delta^*(\cdot \mid \Gamma(t))$ is continuous on $A(t)$. So the arguments of [2, III.33 and III.34] remain valid and $\Gamma(t) = \Theta(t)$ a.e.

(3) First we have to prove that

$$\Sigma(t) = \overline{\operatorname{co} \Gamma(t) + [\mathscr{H}(t)]^0}$$

is measurable. It follows from Theorem 0 that $t \mapsto [\mathscr{H}(t)]^0 = C(t)$ has a closed graph in $T \times E'_c$, where E'_c denotes E' endowed with the topology of uniform convergence on compact subsets of E. It is well known that E'_c is a Lusin space, so $\mathscr{B}(E'_c) = \mathscr{B}(E'_s)$. Hence C is measurable. From dense sequences of measurable selections of Γ and C it is easy (as in [2, III.40]) to deduce a dense sequence of measurable selections of Σ. So if $\dim E < \infty$, Σ is measurable by [2, III.30]. If K is a closed convex LCNL subset of E' such that $\Gamma(t) \subset K$ a.e. then $\mathscr{H}$ contains the constant functions with values in $\operatorname{dom} \delta^*(\cdot \mid K)$ and so $[\mathscr{H}(t)]^0 \subset [\operatorname{dom} \delta^*(\cdot \mid K)]^0 = \operatorname{As} K$. Therefore $\Sigma(t) \subset K$. Then the measurability of Σ follows from [2, III.37, p. 84]. Denote $\mathscr{H}_\Sigma = \operatorname{dom} \delta^*(\cdot \mid L_\Sigma{}^1)$. From $\Sigma(t) \supset \Gamma(t)$ follows $\mathscr{H}_\Sigma \subset \mathscr{H}$. Moreover

$$\delta^*(\varphi(t) \mid \Sigma(t)) = \delta^*(\varphi(t) \mid \Gamma(t)) + \begin{cases} 0 & \text{if} \quad \varphi(t) \in \overline{\mathscr{H}(t)} \\ +\infty & \text{if} \quad \varphi(t) \notin \overline{\mathscr{H}(t)}. \end{cases}$$

So $\varphi \in \mathscr{H} \Rightarrow \varphi \in \mathscr{H}_\Sigma$. Finally $\mathscr{H} = \mathscr{H}_\Sigma$. Hence $\Sigma(t) \supset [\mathscr{H}(t)]^0 = [\mathscr{H}_\Sigma(t)]^0$. The second part of the present theorem ensures that $L_\Sigma{}^1$ is closed. Let us denote by Σ_0 the multifunction such that $\overline{L_\Gamma{}^1} = L^1_{\Sigma_0}$. It is clear that $\Sigma_0(t) \supset \operatorname{co} \Gamma(t)$ a.e., and by Theorem 0 and Remark 2, $\operatorname{As} \Sigma_0(t) = [\mathscr{H}(t)]^0$ a.e. hence $\Sigma_0(t) \supset \Sigma(t)$ a.e. From $L_\Gamma{}^1 \subset L_\Sigma{}^1 \subset L^1_{\Sigma_0} = \overline{L_\Gamma{}^1}$ and the closedness of $L_\Sigma{}^1$ it follows $\overline{L_\Gamma{}^1} = L_\Sigma{}^1$.

Application of Theorem 3 *to the Examples*

EXAMPLE 1. One has for $\varphi \in \mathscr{C}$

$$\delta^*(\varphi \mid L_\Gamma{}^1) = \int_{]0,1]} \delta^* \left(\varphi(t) \,\middle|\, \left[0, \frac{1}{t}\right]\right) dt.$$

It is clear that $\varphi(0) > 0 \Rightarrow \delta^*(\varphi \mid L_\Gamma{}^1) = +\infty$ and that $\varphi(0) < 0 \Rightarrow \delta^*(\varphi \mid L_\Gamma{}^1) < \infty$. Then $\{\varphi \mid \varphi(0) < 0\} \subset \mathscr{H} \subset \{\varphi \mid \varphi(0) \leqslant 0\}$. So $]-\infty, 0[\subset \mathscr{H}(0) \subset]-\infty, 0]$ and $\forall t > 0$, $\mathscr{H}(t) = \mathbf{R}$, hence $[\mathscr{H}(0)]^0 = [0, \infty[$ and $\forall t > 0 [\mathscr{H}(t)]^0 = \{0\}$.

It follows from part (3) of Theorem 3 that

$$\begin{aligned} \Sigma(t) &= [0, \infty[& \text{if} \quad & t = 0 \\ &= [0, 1/t] & \text{if} \quad & t > 0. \end{aligned}$$

EXAMPLE 2. Here if $\varphi \leqslant 0$, $\delta^*(\varphi \mid L_\Gamma{}^1) < \infty$. Otherwise there exists t such that $\varphi(t) > 0$ but as $T - K$ is open and dense in T one has

$\int \delta^*(\varphi(t) \mid \Gamma(t))\, dt = +\infty$. So $\mathscr{H} = -\mathscr{C}_+$, hence $\forall t$, $\mathscr{H}(t) =]-\infty, 0]$ and $\Sigma(t) = [0, \infty[$.

Now we can develop the idea of Remark 5. Consider the two conditions

(C) There exists a closed convex-valued multifunction Γ' such that $\Gamma'(t) = \Gamma(t)$ a.e. and $\forall t$, $\overline{\mathscr{H}(t)} \supset \overline{\operatorname{dom} \delta^*(\cdot \mid \Gamma'(t))}$.

(C′) There exists a closed convex-conical-valued multifunction Γ' such that $\Gamma'(t) = \Gamma(t)$ a.e. and the multifunction $t \to [\Gamma'(t)]^0$ is l.s.c.

COROLLARY 4. (1) (C) *is a necessary condition for* $L_\Gamma{}^1$ *to be closed. Moreover if* $L_\Gamma{}^1$ *is closed one can choose* Γ' *in* (C) *such that* $\forall t$, $\overline{\mathscr{H}(t)} = \overline{\operatorname{dom} \delta^*(\cdot \mid \Gamma'(t))}$.

(2) *Suppose* $\dim E < \infty$ *or* $\Gamma(t)$ *is* a.e. LCNL. *Then* (C) *is a sufficient condition for* $L_\Gamma{}^1$ *to be closed.*

(3) *(C′) is a sufficient condition for* $L_\Gamma{}^1$ *to be closed.*

Remark 6. The relation $\overline{\mathscr{H}(t)} = \overline{\operatorname{dom} \delta^*(\cdot \mid \Gamma'(t))}$ implies that $t \mapsto \overline{\operatorname{dom} \delta^*(\cdot \mid \Gamma'(t))}$ is l.s.c. Indeed if U is an open set then

$$\{t \mid \overline{\mathscr{H}(t)} \cap U \neq \varnothing\} = \{t \mid \mathscr{H}(t) \cap U \neq \varnothing\}$$
$$= \cup\{\varphi^{-1}(U) \mid \varphi \in \mathscr{H}\}$$

is open. But if Γ is not conical valued, the l.s. continuity of $t \mapsto \overline{\operatorname{dom} \delta^*(\cdot \mid \Gamma'(t))}$ is not sufficient to ensure $L_\Gamma{}^1$ to be closed. Indeed in Example 1, $\forall t$, $\operatorname{dom} \delta^*(\cdot \mid \Gamma(t)) = \mathbf{R}$.

Remark 7. Rockafellar ([11, Corollary 5B, p. 466] proves that $L_\Gamma{}^1$ is closed (and also that $\forall t$, As $\Gamma(t) = C(t)$) under the following hypotheses:

(a) $\forall t$, $\Gamma(t)$ is LCNL.

(b) $t \mapsto \operatorname{dom} \delta^*(\cdot \mid \Gamma(t))$ is fully l.s.c. (that is a stronger assumption than l.s.c.-ness).

(c) $\forall V$ open subset of T, $\forall x \in E = \mathbf{R}^n$ such that $\exists$ a neighborhood U of x such that $\forall t \in V$, $\operatorname{dom} \delta^*(\cdot \mid \Gamma(t)) \supset U$, then

$$\int_V \mid \delta^*(x \mid \Gamma(t)) \mid \mu(dt) < \infty.$$

(d) the support of μ is T.

Proof. (1) By Theorem 3(1), As $\Gamma(t) = [\mathscr{H}(t)]^0$ a.e. By [2, I.7, p. 8] As $\Gamma(t) = [\operatorname{dom} \delta^*(\cdot \mid \Gamma(t))]^0$ hence $\overline{\operatorname{dom} \delta^*(\cdot \mid \Gamma(t))} = \overline{\mathscr{H}(t)}$ a.e. We choose

$$\Gamma'(t) = \Gamma(t) \qquad \text{if} \qquad \text{As}\, \Gamma(t) = [\mathscr{H}(t)]^0$$
$$= [\mathscr{H}(t)]^0 \qquad \text{otherwise.}$$

So we have $\forall t$, $\overline{\operatorname{dom} \delta^*(\cdot \mid \Gamma'(t))} = \overline{\mathscr{H}(t)}$. That is clear in the first eventuality, and in the second it follows from the fact that $\mathscr{H}(t)$ is conic.

(2) One has

$$\text{As } \Gamma'(t) = [\operatorname{dom} \delta^*(\cdot \mid \Gamma'(t))]^0 \supset [\mathscr{H}(t)]^0.$$

So by Theorem 3(2), $L_\Gamma{}^1$ is closed.

(3) Let $\mathscr{K}$ denote the set of continuous selections of $t \mapsto [\Gamma'(t)]^0$. As T is compact and E Fréchet, the Michael theorem [5] ensures $\mathscr{K}(t) = [\Gamma'(t)]^0$. Let (φ_n) be a dense sequence in $\mathscr{K}$ (for the uniform convergence topology). Then

$$\forall t, \overline{\{\varphi_n(t) \mid n \in \mathbf{N}\}} = [\Gamma'(t)]^0.$$

So one has

$$\Gamma'(t) = \{x' \mid \forall n, \langle x', \varphi_n(t)\rangle \leqslant 0\},$$

and by Theorem 1, $L^1_{\Gamma'}$ is closed.

6. Functional Results

We set again some notation and hypotheses.

Let $f\colon T \times E' \to \overline{\mathbf{R}}$ be an integrand, that is a measurable function. We shall write f_t for $f(t, \cdot)$. We define $I_f(u)$ for $u \in \mathscr{L}^1_{E'}$ by

$$I_f(u) = \int f(t, u(t))\, \mu(dt)$$

with the convention that if both the negative and positive parts of $f(t, u(t))$ are nonintegrable, then $I_f(u) = +\infty$.

To f corresponds the multifunction $\Gamma(t) = \operatorname{epi} f_t$ with values in $E' \times \mathbf{R}$. By [2, VII.1, p. 196] Γ is measurable. The hypotheses (H1) and (H2) relative to Γ become

(H′1) μ is nonatomic or f_t is a.e. convex.

(H′2) $\exists u \in L^1_{E'}$ such that $I_f(u) < \infty$ ($I_f(u) = -\infty$ is allowed).

Indeed

$$\begin{aligned} (\text{H2}) &\Leftrightarrow \exists (u, v) \in L_\Gamma{}^1 \\ &\Leftrightarrow \exists u \in L^1_{E'}, \quad \exists v \in L_{\mathbb{R}}{}^1 \quad \text{such that} \quad f(t, u(t)) \leqslant v(t) \quad \text{a.e.} \\ &\Leftrightarrow (\text{H}'2). \end{aligned}$$

Under these hypotheses we associate to Γ the set

$$\mathscr{H} = \operatorname{dom} \delta^*(\cdot \mid L_\Gamma{}^1) = \{(\varphi, \alpha) \in \mathscr{C}_{E\times\mathbb{R}} \mid \delta^*(\varphi, \alpha) \mid L_\Gamma{}^1) < \infty\},$$

and the multifunctions Σ and C defined in Theorem 0.

It follows from $\Sigma(t) \supset \Gamma(t)$ a.e. that $\Sigma(t)$ is a.e. the epigraph of a convex l.s.c. function g_t. By [2, VII.1, p. 196] the function $g\colon T \times E' \to \bar{\mathbf{R}}$ defined by $g(t, x') = g_t(x')$ is an integrand. It will be proved in Lemma 8 that $C(t)$ is also an epigraph. Consider $f^*\colon T \times E \to \bar{\mathbf{R}}$ defined by $f^*(t, \cdot) = [f(t, \cdot)]^*$. We define $I_{f^*}(\varphi)$ for $\varphi \in \mathscr{C}_E$ by

$$I_{f^*}(\varphi) = \int f^*(t, \varphi(t))\, \mu(dt).$$

Thanks to a famous theorem of Rockafellar [2, VII.7, p. 200], under (H′2)

$$I_{f^*}(\varphi) = (I_f)^*\, (\varphi) = \sup\{\langle u, \varphi\rangle - I_f(u) \mid u \in L^1_{E'}\}.$$

Let

$$\mathscr{G} = \operatorname{dom} I_{f^*} = \{\varphi \in \mathscr{C}_E \mid I_{f^*}(\varphi) < \infty\}.$$

We assume the following hypothesis

(H3) $\mathscr{G} \neq \phi$.

That means $\exists\varphi_0 \in \mathscr{C}_E$ such that $\int f^*(t, \varphi_0(t))\, \mu(dt) < \infty \Leftrightarrow \exists\varphi_0 \in \mathscr{C}_E$ $\exists\psi \in \mathscr{L}_{\mathbb{R}}{}^1$ such that $f^*(t, \varphi_0(t)) \leqslant \psi(t)$ a.e. $\Leftrightarrow \exists\varphi_0 \in \mathscr{C}_E$, $\exists\psi \in \mathscr{L}_{\mathbb{R}}{}^1$ such that for almost every t,

$$\forall x', \langle x', \varphi_0(t)\rangle - f(t, x') \leqslant \psi(t).$$

So (H3) is equivalent to

(H′3) $\exists\varphi_0 \in \mathscr{C}_E$, $\exists\psi \in \mathscr{L}_{\mathbb{R}}{}^1$ such that for almost every t,

$$\forall x', f(t, x') \geqslant \langle x', \varphi_0(t)\rangle - \psi(t).$$

Remark 8. If I_f is a convex proper $\sigma(L^1_{E'}, \mathscr{C}_E)$ l.s.c. functional, then its polar I_{f^*} is also proper, so (H3) is satisfied. Moreover under (H′1) it will be proved in Lemma 6 that if I_f is l.s.c. it is necessarily convex.

The following hypothesis is assumed on and after Lemma 8:

(H4) The support of μ is T.

Lemma 5. *Let f be an integrand. Let $A\colon L^1_{E'\times\mathbb{R}} \to L^1_{E'} \times \mathbf{R}$ be defined by $A(u, v) = (u, \int v\, d\mu)$. Then* $\operatorname{epi} I_f = A(L_\Gamma{}^1)$.

Proof. Let $u \in \mathscr{L}^1_{E'}$, $r \in \mathbf{R}$. Then

$$(u, r) \in \operatorname{epi} I_f \Leftrightarrow r \geqslant I_f(u)$$
$$\Leftrightarrow \exists v \in \mathscr{L}_{\mathbb{R}}^1$$

such that

$$v(t) \geqslant f(t, u(t)) \quad \text{a.e.} \qquad \text{and} \qquad \int v \, d\mu = r$$

(we prove $\Rightarrow$ below)

$$\Leftrightarrow \exists v \in \mathscr{L}_{\mathbb{R}}^1 \qquad \text{such that} \qquad (u, v) \in \mathscr{L}_\Gamma^1 \quad \text{and} \quad A(u, v) = (u, r).$$

We prove now the second $\Rightarrow$. Thanks to the convention in the definition of I_f,

$$r \geqslant I_f(u) \Rightarrow f(t, u(t))^+ \text{ is integrable.}$$

If $f(t, u(t))^-$ is also integrable we put

$$v(t) = f(t, u(t)) + \frac{r - I_f(u)}{\mu(T)}.$$

If $f(t, u(t))^-$ is not integrable, then for every $n \geqslant 0$ $h_n(t) = \sup(-n, f(t, u(t))$ is integrable. By the Fatou lemma, for n large enough, one has $\int h_n \leqslant r$. So we put

$$v = h_n + \frac{r - \int h_n \, d\mu}{\mu(T)}. \quad \blacksquare$$

From now on we assume (H′1), (H′2), and (H3).

LEMMA 6. *The* l.s.c. *hull of* I_f *is convex and its epigraph is* $\overline{\operatorname{epi} I_f}$. *The polar of* I_{f*} *within the duality* $\mathscr{C}_E$, $\mathscr{M}_{E'}$ *is convex and its epigraph is* $\overline{\overline{\operatorname{epi} I_f}}$.

Proof. As A is continuous, the formula

$$\overline{\operatorname{epi} I_f} = \overline{A(L_\Gamma^1)} \qquad \text{implies} \qquad \overline{\operatorname{epi} I_f} = \overline{A(\overline{L_\Gamma^1})}.$$

The Liapunov theorem implies that $\overline{L_\Gamma^1}$ is convex. So the first assertion is proved.

(2) Let $\hat{I}_f \colon \mathscr{M}_{E'} \to \overline{\mathbf{R}}$ be defined by

$$\begin{aligned} \hat{I}_f(m) &= I_f(u) && \text{if} \quad m = u \cdot \mu \\ &= +\infty && \text{if} \quad m \text{ is not absolutely continuous with respect to } \mu. \end{aligned}$$

We said above that $I_{f*} = (I_f)^*$. One has also

$$I_{f*} = (\hat{I}_f)^* \qquad \text{within the duality} \qquad \mathscr{M}_{E'}, \mathscr{C}_E.$$

Moreover $\hat{I}_f$ is greater than the following continuous affine function

$$m \mapsto \langle m, \varphi_0 \rangle - \int \psi \, d\mu \qquad \text{(thanks to } (H'3)).$$

So by a well known result [2, I.5, p. 6] the greatest convex l.s.c. function less than $\hat{I}_f$ is its bipolar $(\hat{I}_f)^{**} = (I_{f^*})^*$. Hence

$$\begin{aligned} \operatorname{epi}(I_{f^*})^* &= \overline{\overline{\operatorname{co}\operatorname{epi}\hat{I}_f}} = \overline{\overline{\operatorname{co}\operatorname{epi} I_f}} \\ &= \overline{\overline{\operatorname{epi} I_f}} \qquad \text{since} \qquad \overline{\operatorname{epi} I_f} \text{ is convex.} \quad \blacksquare \end{aligned}$$

We need now the adjoints A^* and A^{**}. They are easy to calculate:

$$A^* : \mathscr{C}_E \times \mathbf{R} \to \mathscr{C}_{E \times \mathbb{R}} = \mathscr{C}_E \times \mathscr{C}_{\mathbb{R}}$$

satisfies

$$A^*(\varphi, r) = (\varphi, r\chi_T),$$

and

$$A^{**} : \mathscr{M}_{E'} \times \mathscr{M}_{\mathbb{R}} \to \mathscr{M}_{E'} \times \mathbf{R}$$

satisfies

$$A^{**}(\lambda, \sigma) = (\lambda, \sigma(T)).$$

LEMMA 7. *One has* $\overline{\overline{\operatorname{epi} I_f}} = A^{**}(\overline{\overline{L_\Gamma^1}})$.

Proof. From Lemma 5 and the continuity of A^{**} follows $\overline{\overline{\operatorname{epi} I_f}} \supset A^{**}(\overline{\overline{L_\Gamma^1}})$. Conversely let $(\lambda, r) \in \overline{\overline{\operatorname{epi} I_f}}$.

There exists a generalized sequence $(u_i, r_i)_{i \in I}$ in $\operatorname{epi} I_f$ which converges to (λ, r). There exists $v_i \in L_{\mathbb{R}}^1$ such that $(u_i, v_i) \in L_\Gamma^1$ and $\int v_i \, d\mu = r_i$.

By (H'3) $f(t, x') \geqslant \langle x', \varphi_0(t) \rangle - \psi(t)$. So

$$-v_i(t) + \langle u_i(t), \varphi_0(t) \rangle \leqslant -f(t, u_i(t)) + \langle u_i(t), \varphi_0(t) \rangle \leqslant \psi(t).$$

One has

$$\begin{aligned} \int (-v_i + \langle u_i, \varphi_0 \rangle)^- \, d\mu &= \int (-v_i + \langle u_i, \varphi_0 \rangle)^+ \, d\mu - \int (-v_i + \langle u_i, \varphi_0 \rangle) \, d\mu \\ &\leqslant \int \psi^+ \, d\mu - \int (-v_i + \langle u_i, \varphi_0 \rangle) \, d\mu \end{aligned}$$

and

$$\int (-v_i + \langle u_i, \varphi_0 \rangle) \, d\mu \to -r + \langle \lambda, \varphi_0 \rangle.$$

Hence for i large enough, say $i \geqslant i_0$ $\int (-v_i + \langle u_i, \varphi_0 \rangle)^- \, d\mu$ is bounded and so $\int | -v_i + \langle u_i, \varphi_0 \rangle | \, d\mu$ is also bounded.

The real measures $(-v_i + \langle u_i, \varphi_0\rangle)\,\mu$ for $i \geqslant i_0$ belongs to a weakly compact subset of $\mathscr{M}_{\mathbb{R}}$. Let $\mathscr{U}$ be an ultrafilter on I finer than the filter of sections, the limit $\lim_{\mathscr{U}}(-v_i + \langle u_i, \varphi_0\rangle)\,\mu$ exists. It is easy to see that $\langle u_i, \varphi_0\rangle\,\mu$ converges to the real Radon measure $\alpha \mapsto \langle \lambda, \alpha\varphi_0\rangle$, hence $\lim_{\mathscr{U}} v_i\mu$ exists. Thanks to the continuity of A^{**} we have

$$(\lambda, r) = A^{**}(\lambda, \lim_{\mathscr{U}} v_i\mu),$$

where $(\lambda, \lim_{\mathscr{U}} v_i\mu) = \lim_{\mathscr{U}}(u_i, v_i)$ belongs to $\overline{\overline{L_\Gamma{}^1}}$.

Lemma 8. *For every t, $C(t) = \operatorname{epi} \delta^*(\cdot \mid \mathscr{G}(t))$ (where $\mathscr{G}(t) = \{\varphi(t) \mid \varphi \in \mathscr{G}\}$).*

Proof. We have to prove $[\mathscr{H}(t)]^0 = \operatorname{epi} \delta^*(\cdot \mid \mathscr{G}(t))$. First remark that

$$\begin{aligned} \varphi \in \mathscr{G} &\Leftrightarrow \int f^*(t, \varphi(t))\,\mu(dt) < \infty \\ &\Leftrightarrow \int \delta^*((\varphi(t), -1) \mid \operatorname{epi} f_t)\,\mu(dt) < \infty \\ &\Leftrightarrow \delta^*((\varphi, -\chi_T) \mid L_\Gamma{}^1) < \infty \\ &\Leftrightarrow (\varphi, -\chi_T) \in \mathscr{H}. \end{aligned}$$

If $(x', r) \in [\mathscr{H}(t)]^0$, for every $\varphi \in \mathscr{G}$, one has $\langle (x', r), (\varphi(t), -1)\rangle \leqslant 0$ hence $r \geqslant \langle x', \varphi(t)\rangle$ which implies $(x', r) \in \operatorname{epi} \delta^*(\cdot \mid \mathscr{G}(t))$. It remains to prove $\operatorname{epi} \delta^*(\cdot \mid \mathscr{G}(t)) \subset [\mathscr{H}(t)]^0$. Let $(x', r) \in \operatorname{epi} \delta^*(\cdot \mid \mathscr{G}(t))$ and $(\varphi, \alpha) \in \mathscr{H}$. We have to prove $\langle (x', r), (\varphi(t), \alpha(t))\rangle \leqslant 0$.

First let us prove $\forall s$, $\alpha(s) \leqslant 0$. One has

$$\delta^*((\varphi, \alpha) \mid L_\Gamma{}^1) = \int \delta^*((\varphi(s), \alpha(s)) \mid \operatorname{epi} f_s)\,\mu(ds) < \infty.$$

As $\operatorname{epi} f_s \neq \phi$ a.e., $\alpha(s)$ is $\leqslant 0$ a.e. But α is a continuous function and the support of μ is T, so $\forall s$, $\alpha(s) \leqslant 0$.

(a) The simplest case is when $\forall s$, $\alpha(s) < 0$. Therefore $-1/\alpha$ is a positive continuous, hence bounded function. So

$$\begin{aligned} \delta^*\left(-\frac{1}{\alpha}(\varphi, \alpha) \mid L_\Gamma{}^1\right) &= \int \delta^*\left(-\frac{1}{\alpha}(\varphi, \alpha) \mid \Gamma(\cdot)\right) d\mu \\ &= \int -\frac{1}{\alpha}\,\delta^*((\varphi, \alpha) \mid \Gamma(\cdot))\,d\mu \end{aligned}$$

is finite.

Hence $-(1/\alpha)(\varphi, \alpha) = (-\varphi/\alpha, -\chi_T)$ belongs to $\mathscr{H}$, and $-\varphi/\alpha \in \mathscr{G}$. Therefore $r \geqslant \langle x', -\varphi(t)/\alpha(t)\rangle$ or $\langle x', \varphi(t)\rangle + r\alpha(t) \leqslant 0$.

(b) Consider now the general case. Let $\varphi_0 \in \mathscr{G}$. For every $\lambda \in [0, 1[$ $(\psi, \beta) = \lambda(\varphi, \alpha) + (1 - \lambda)(\varphi_0, -\chi_T)$ belongs to $\mathscr{H}$ and $\forall s, \beta(s) < 0$. So by (a)

$$\langle x', \psi(t)\rangle + r\beta(t) \leqslant 0.$$

As λ can converge to 1, one has $\langle x', \varphi(t)\rangle + r\alpha(t) \leqslant 0$. ∎

The Lebesgue decomposition theorem is valid in $\mathscr{M}_{E'}$ ([13, Remark 2 after Theorem 1]. If $m \in \mathscr{M}_{E'}$ one has $m = (dm/d\mu)\,\mu + m_s$, where m_s is μ-singular. There exists a real bounded positive measure θ such that m_s is absolutely continuous with respect to θ. Then $m = (dm/d\mu) + (dm_s/d\theta)\,\theta$.

THEOREM 9. *With the foregoing notation one has*

$$(I_{f*})^*(m) = I_g\left(\frac{dm}{d\mu}\right) + \int \delta^*\left(\frac{dm_s}{d\theta}(t) \mid \mathscr{G}(t)\right)\theta(dt). \tag{6}$$

Remark 9. The last term does not depend on the measure θ.

Proof. (1) First we show that the right-hand side is meaningful. We have noticed in the proof of part 3 of Theorem 3 that C is measurable, then by Lemma 8 and [2, VIII.1, p. 196], $(t, x') \mapsto \delta^*(x' \mid \mathscr{G}(t))$ is an integrand.

Let $\varphi_0 \in \mathscr{G}$, then

$$\int \delta^*\left(\frac{dms}{d\theta}(t) \mid \mathscr{G}(t)\right)\theta(dt) \geqslant \int \left\langle \frac{dms}{d\theta}, \varphi_0\right\rangle d\theta > -\infty.$$

Let φ_0 and ψ satisfying (H′3). Consider

$$\Gamma_1(t) = \operatorname{epi}(x' \mapsto \langle x', \varphi_0(t)\rangle - \psi(t)).$$

By Theorem 1, $L^1_{\Gamma_1}$ is closed, so that $\Sigma(t) \subset \Gamma_1(t)$ a.e., that is,

$$g(t, x') \geqslant \langle x', \varphi_0(t)\rangle - \psi(t).$$

Therefore $I_g(dm/d\mu) > -\infty$

(2) By Lemma 7

$$(m, r) \in \overline{\overline{\operatorname{epi} I_f}} \Leftrightarrow \exists (m, \sigma) \in \overline{\overline{L_\Gamma^1}} \qquad \text{such that} \qquad \sigma(T) = r.$$

By Lemma 6

$$\begin{aligned}(I_{f*})^*(m) &= \inf\{r \mid (m, r) \in \overline{\overline{\operatorname{epi} I_f}}\}\\ &= \inf\{\sigma(T) \mid (m, \sigma) \in \overline{\overline{L_\Gamma^1}}\}.\end{aligned}$$

Let $(m, \sigma) \in \overline{\overline{L_\Gamma^1}}$. We shall prove

$$\sigma(T) \geqslant I_g\left(\frac{dm}{d\mu}\right) + \int \delta^*\left(\frac{dm_s}{d\theta}(t) \mid \mathscr{G}(t)\right) \theta(dt). \tag{7}$$

Let (m_s, σ_s) be the μ-singular part of (m, σ). By Theorem 0 there exists a real positive measure θ singular with respect to μ such that (m_s, σ_s) is absolutely continuous with respect to θ and

$$\left(\frac{dm_s}{d\theta}, \frac{d\sigma_s}{d\theta}\right) \in L_C^1(\theta), \tag{8}$$

$$\left(\frac{dm}{d\mu}, \frac{d\sigma}{d\mu}\right) \in L_\Sigma^1. \tag{9}$$

From (8) follows $(d\sigma_s/d\theta)(t) \geqslant \delta^*((dm_s/d\theta)(t) \mid \mathscr{G}(t))$ θ-a.e. so that $\sigma_s(T) \geqslant \int \delta^*(dm_s/d\theta)(t) \mid \mathscr{G}(t))\, \theta(dt)$.

From (9) follows $(d\sigma/d\mu)(t) \geqslant g(t, (dm/d\mu)(t))$ μ a.e. so that $\int (d\sigma/d\mu)\, d\mu \geqslant I_g(dm/d\mu)$. As $\sigma(T) = \int (d\sigma/d\mu)\, d\mu + \sigma_s(T)$, inequality (7) is proved.

(3) To prove (6) it remains to show that if the right-hand side is finite, there exists σ such that $(m, \sigma) \in \overline{\overline{L_\Gamma^1}}$ and (7) becomes an equality. That is true because it suffices to put

$$\sigma = g\left(\cdot, \frac{dm}{d\mu}(\cdot)\right)\mu + \delta^*\left(\frac{dm_s}{d\theta}(\cdot) \mid \mathscr{G}(\cdot)\right)\theta.$$

Corollary 10. *The* l.s.c. *hull of* I_f *is* I_g. *If* $\dim E < \infty$ *or if* $\operatorname{epi} f_t$ *is contained in a fixed closed convex* LCNL *subset of* $E' \times \mathbf{R}$ *one has*

$$\operatorname{epi} g_t = \overline{\operatorname{co} \operatorname{epi} f_t + \operatorname{epi} \delta^*(\cdot \mid \mathscr{G}(t))}.$$

Proof. By Lemma 6 the l.s.c. hull of I_f is equal to $(I_{f*})^*$ restricted to $L^1_{E'}$. By Theorem 9 it is I_g. The second assertion follows from Theorem 3 (3).

Corollary 11. *Suppose* $\dim E < \infty$ *or* $\operatorname{epi} f_t$ *is* a.e. LCNL. *The following condition is sufficient for I_f to be* l.s.c.: f_t *is* a.e. *convex* l.s.c. *and* $\operatorname{rec} f_t \leqslant \delta^*(\cdot \mid \mathscr{G}(t))$ a.e. (*where* $\operatorname{rec} f_t$ *is defined by* $\operatorname{epi}(\operatorname{rec} f_t) = \operatorname{As}(\operatorname{epi} f_t)$).

Proof. The condition $\operatorname{rec} f_t \leqslant \delta^*(\cdot \mid \mathscr{G}(t))$ means $\operatorname{As} \Gamma(t) \supset C(t)$. So by Theorem 3(2), L_Γ^1 is closed, hence $\Gamma(t) = \Sigma(t)$ a.e. that is, $f_t = g_t$ a.e. By Corollary 10 I_f is l.s.c.

Corollary 12. *The following condition is necessary for I_f to be* l.s.c.: f_t *is* a.e. *convex* l.s.c. *and* $\operatorname{rec} f_t = \delta^*(\cdot \mid \mathscr{G}(t))$ a.e. (*where* $\operatorname{rec} f_t$ *is defined by* $\operatorname{epi}(\operatorname{rec} f_t) = \operatorname{As}(\operatorname{epi} f_t)$).

Proof. If I_f is l.s.c. it follows from Corollary 10 that $I_f = I_g$. Thanks to the next lemma, $f_t = g_t$ a.e. So $L_\Gamma{}^1 = L_\Sigma{}^1$ is closed, and the necessary condition follows from Theorem 3(1).

LEMMA 13. *Let f and g be two integrands such that:*

(a) $g_t \leqslant f_t$ a.e.

(b) $\exists u \in L^1_{E'}$ *such that* $I_f(u) < \infty$,

(c) $\forall u \in L^1_{E'}$, $f(t, u(t))^-$ *is integrable,*

(d) $I_f = I_g$.

Then $f_t = g_t$ a.e.

Proof. Let $\Gamma(t) = \operatorname{epi} f_t$, $\Sigma(t) = \operatorname{epi} g_t$. By (a) $\Sigma(t) \supset \Gamma(t)$ a.e. By (b) $L_\Gamma{}^1 \neq \phi$. By Lemma 2 if $f_t = g_t$ a.e. is not true one has $L_\Gamma{}^1 \neq L_\Sigma{}^1$. So there exists $(u, v) \in L_\Sigma{}^1 - L_\Gamma{}^1$.

Put $v'(t) = \inf(v(t), f(t, u(t)))$.

Thanks to (c) v' is still integrable. The function v' has the following properties.

(1) $g(t, u(t)) \leqslant v'(t) \leqslant f(t, u(t))$,

(2) $v'(t) < f(t, u(t))$ on a nonnegligible set, because $(u, v) \notin L_\Gamma{}^1$.

So we have $I_g(u) \leqslant \int v' \, d\mu < I_f(u)$ which is a contradiction to (d).

Remark 10. In Olech [7], $E = \mathbf{R}^n$, so our Corollaries 11 and 12 contain his characterization: I_f is l.s.c. iff $L_\Gamma{}^1$ is closed.

EXAMPLE 3. Let $T = [0, 1]$, $\mu = dt + \delta_0$, $E = \mathbf{R}$

$$\begin{aligned} f_t &= x \mapsto x^2 & &\text{if} \quad t = 0 \\ &= x \mapsto t^2 x^2 & &\text{if} \quad t > 0. \end{aligned}$$

Then I_f is not l.s.c. because if $u_n = n\chi_{]0,1/n]}$ one has $u_n \to \chi_{\{0\}}$, but

$$I_f(u_n) = \int_0^{1/n} t^2 n^2 \, dt = \frac{1}{3n} \to 0 \qquad \text{and} \qquad I_f(\chi_{\{0\}}) = 1.$$

We can calculate g. Indeed

$$\begin{aligned} f_t^* &= x' \mapsto \tfrac{1}{4} x'^2 & &\text{if} \quad t = 0 \\ &= x' \mapsto \frac{1}{4t^2} x'^2 & &\text{if} \quad t > 0. \end{aligned}$$

So if $\varphi \in \mathscr{C}_\mathbb{R}$ one has $I_{f^*}(\varphi) < \infty \Rightarrow \varphi(0) = 0$.

Hence $\mathscr{G}(0) = \{0\}$ and it is clear that $\mathscr{G}(t) = \mathbf{R}$ for $t > 0$. By Corollary 10

$$\begin{aligned} g_t &= 0 && \text{if} \quad t = 0 \\ &= x \mapsto t^2x^2 && \text{if} \quad t > 0. \end{aligned}$$

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